Name:______________________________

MA 4933/6933 Section 01 *Practice Exam 1* *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. Let $C[a,b]$ be the space of continuous functions. Prove that $\|\cdot\|_1$ defines a norm on $C[a,b]$ where

$$\|f\|_1 = \int_a^b |f(x)|\, dx$$

and show that $\|f\|_1 \leq \|f\|_\infty (b-a)$.

2. Let (S,d) be a metric space. Prove that if $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x,y)$.

3. Let c_{00} be the space of all sequences that are eventually zero. Prove that c_{00} is a dense subspace of $\ell^p(\mathbb{N})$ for $p < \infty$ where

$$\ell^p(\mathbb{N}) = \left\{ a = \{a_n\} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\} .$$

4. Let c_0 be the space of all convergent sequences with limit zero. Prove that c_0 is a closed subspace of $\ell^\infty(\mathbb{N})$ and that $c_0 = \overline{c_{00}}$ with respect to $\| \cdot \|_\infty$.

5. Let $C[0,1]$ be the space of continuous functions equipped with $\| \cdot \|_\infty$ and let $g \in C[0,1]$. Define the operator $T : C[0,1] \to C[0,1]$ by $T(f) = fg$. Prove that T is linear, bounded and compute $\|T\|$.

6. Let $\ell^p(\mathbb{N})$ be defined as above. Define the right shift operator $T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ by

$$T(\{x_1, x_2, \ldots\}) = \{0, x_1, x_2, \ldots\} \ .$$

Prove that T is linear, bounded and compute $\|T\|$.

7. Let (S, d) be a metric space with d being the distance-1 metric, that is

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} .$$

Characterize the Cauchy sequences in this space. Is this a complete space? (Be sure to justify.)

8. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $|f'(t)| \leq \alpha$, where $0 < \alpha < 1$. Prove that f is a contraction.